Dynamical systems method (DSM) for unbounded operators

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Abstract

Let L be an unbounded linear operator in a real Hilbert space H, a generator of C_0 semigroup, and $g : H \to H$ be a C^2_{loc} nonlinear map. The DSM (dynamical systems method) for solving equation $F(v) := Lv + gv = 0$ consists of solving the Cauchy problem $\dot{u} = \Phi(t, u)$, $u(0) = u_0$, where Φ is a suitable operator, and proving that i) $\exists u(t) \quad \forall t > 0$, ii) $\exists u(\infty)$, and iii) $F(u(\infty)) = 0$.

Conditions on L and g are given which allow one to choose Φ such that i), ii), and iii) hold.

1 Introduction

Let H be a real Hilbert space, L be a linear, densely defined in H, closed operator, a generator of C_0 semigroup (see[1]), $g : H \to H$ be a nonlinear C^2_{loc} map, i.e.,

$$\sup_{u \in B(u_0,R)} \|g^{(j)}(u)\| \le m_j, \quad j = 0, 1, 2, \quad B(u_0, R) := \{u : \|u - u_0\| \le R\}, \tag{1.1}$$

where $g^{(j)}$ is the Fréchet derivative of order j, $R > 0$ is some number, and $u_0 \in H$ is some element. In many applications the problems can be formulated as the following operator equation:

$$F(v) := Lv + g(v) = 0. \tag{1.2}$$

We want to study this equation by the dynamical sysstems method (DSM), which allows one also to develop numerical methods for solving equation (1.2). The DSM for solving

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equation (1.2) consists of solving the problem:

$$\dot{u} = \Phi(t, u), \quad u(0) = u_0, \tag{1.3}$$

where $\dot{u} := \frac{du}{dt}$, and $\Phi(t, u)$ is a nonlinear operator chosen so that problem (1.3) has a unique global solution which stabilizes at infinity to the solution of equation (1.2):

$$\text{i) } \exists u(t) \forall t > 0, \qquad \text{ii) } \exists u(\infty), \qquad \text{iii) } F(u(\infty)) = 0. \tag{1.4}$$

In [2] the DSM has been studied and justified for $F \in C^2_{loc}$ and

$$\sup_{u \in B(u_0, R)} \|[F'(u)]^{-1}\| \le m_1; \tag{1.5}$$

for monotone $F \in C^2_{loc}$; for monotone hemicontinuous defined on all of H operators F; for non-monotone $F \in C^2_{loc}$ such that there exists a y such that $F(y) = 0$ and the operator $A := F'(y)$ maps any ball $B(0, r)$ centered at the origin and of sufficiently small radius $r > 0$ into a set which has a non-empty intersection with $B(0, R)$; and for $F \in C^2_{loc}$ satisfying a spectral condition: $\|(F'(u) + \varepsilon)^{-1}\| \le (c\varepsilon)^{-1}$, $0 < c \le 1$, in which case F is replaced by $F + \varepsilon I$ in (1.3), and then ε is taken to zero.

In this paper the DSM is justified for a class of nonlinear unbounded operators of the type $L + g$, where L is a generator of a C_0 semigroup, $g \in C^2_{loc}$, and some suitable additional assumptions are made.

Which assumptions are suitable? A simple example is:

$$\|L^{-1}\| \le m. \tag{1.6}$$

If (1.6) holds, then (1.2) is equivalent to

$$f(u) := u + L^{-1} g(u) = 0, \quad f \in C^2_{loc}. \tag{1.7}$$

Assume

$$\sup_{u \in B(u_0, R)} \|[I + L^{-1} g'(u)]^{-1}\| \le m_1. \tag{1.8}$$

This assumption, holds, e.g., if $L^{-1} g'(u)$ is a compact operator in H for any $u \in B(u_0, R)$ and the operator $I + L^{-1} g'(u)$ is injective.

Our first result is the following theorem:

Theorem 1.1. *Assume* (1.6), (1.8), *and let* $\Phi := -[I + L^{-1} g'(u)]^{-1}[u + L^{-1} g(u)]$. *If*

$$\|u_0 + L^{-1} g(u_0)\| m_1 \le R, \tag{1.9}$$

then equation (1.2) *has a unique solution* $v \in B(u_0, R)$, *the conclusions i), ii), and iii) hold for* (1.3), *and* $u(\infty) = v$.

Remark 1.1. *If L is not boundedly invertible, i.e.,* (1.6) *fails, then one can use the following assumption (A):*

Assumption (A). *There exists a sector* $S = \{z : 0 < |z| \le a, \ |\arg z - \pi| \le \delta\}$, *which consists of regular points of* L. *Here* $a > 0$ *and* $\delta > 0$ *are arbitrary small fixed numbers.*

If assumption (A) holds, then

$$\|(L+\varepsilon)^{-1}\| \le \frac{1}{\varepsilon \sin(\delta)}. \tag{1.10}$$

This estimate holds in particular if $L = L^* \ge 0$, and in this case $\sin(\delta) = 1$.

The following theorem is our next result:

Theorem 1.2. *Assume that* $L^* = L \ge 0$ *is a densely defined linear operator,* $g \in C^2_{loc}$, $g'(u) \ge 0 \ \forall u \in H$, *equation* (1.2) *is solvable and* v *is its (unique) minimal-norm solution. Define* $\Phi(u) = -[I + (L+\varepsilon)^{-1} g'(u)]^{-1}[u + (L+\varepsilon)^{-1} g(u)]$, $\varepsilon =$ *const* > 0. *Assume that (1.8) holds with* $L_\varepsilon := L + \varepsilon I$ *replacing* L, *and* $m_1 = m_1(\varepsilon) > 0$. *Then problem* (1.3) *has a unique global solution* $u_\varepsilon(t)$, *there exists* $u_\varepsilon(\infty) := v_\varepsilon$, *and* $F(v_\varepsilon) := L v_\varepsilon + g(v_\varepsilon) = 0$. *Moreover, there exists the limit* $\lim_{\varepsilon \to 0} v_\varepsilon = v$, *which is the unique minimal-norm solution to* (1.2).

In section 2 proofs are given.

2 Proofs

Proof of Theorem 1.1. If $\Phi = -[I + L^{-1} g'(u)]^{-1}[u + L^{-1} g(u)]$, and $p(t) := \|u + L^{-1} g(u)\|$, then $p\dot{p} = -p^2$. Thus

$$p(t) = p(0) e^{-t}. \tag{2.1}$$

From (2.1), (1.8) and (1.3) one gets

$$\|\dot{u}\| \le m_1 p(0) e^{-t}, \quad p(0) = \|u_0 + L^{-1} g(u_0)\|. \tag{2.2}$$

Inequality (2.2) implies the global existence of $u(t)$, the existence of $u(\infty) := \lim_{t\to\infty} u(t)$, and the estimates:

$$\|u(t) - u(\infty)\| \le m_1 p(0) e^{-t}, \quad \|u(t) - u_0\| \le m_1 p(0). \tag{2.3}$$

If (1.9) holds, then (2.3) implies $\|u(t) - u_0\| \le R$, so the trajectory $u(t)$ stays in the ball $B(u_0, R)$, that is, $u(t) \in B(u_0, R) \ \forall t \ge 0$. Passing to the limit $t \to \infty$ in equation (1.3) yields

$$0 = -[I + L^{-1} g'(u(\infty))]^{-1}[u(\infty) + L^{-1} g(u(\infty))]. \tag{2.4}$$

Thus $u(\infty) := v$ solves the equation $v + L^{-1} g(v) = 0$, so v solves (1.2), and therefore i), ii) and iii) hold. Theorem 1.1 is proved. □

Proof of Theorem 1.2. If $L^* = L \geq 0$ in H and $g'(u) \geq 0$, then, for any $\varepsilon > 0$, Theorem 1.1 yields the existence of a unique solution v_ε to equation (1.2) with L replaced by $L + \varepsilon I$. This solution $v_\varepsilon = u_\varepsilon(\infty)$, where $u_\varepsilon(t)$ is the solution to (1.3) with

$$\Phi = -[I + (L+\varepsilon)^{-1}g'(u)]^{-1}[u + (L+\varepsilon)^{-1}g(u)].$$

Let us prove that $\lim_{\varepsilon\to 0} v_\varepsilon = v$, where v solves (1.2). We do not assume that (1.2) has a unique solution.

Let $v_\varepsilon - v := w$. Then $Lw + \varepsilon v_\varepsilon + g(v_\varepsilon) - g(v) = 0$, so, using the assumptions $L \geq 0$ and $g'(u) \geq 0$, one gets $\varepsilon(v_\varepsilon, v_\varepsilon - v) \leq 0$, $\|v_\varepsilon\|^2 \leq \|v_\varepsilon\|\|v\|$, and $\|v_\varepsilon\| \leq \|v\|$, $\forall \varepsilon > 0$. Thus $v_\varepsilon \rightharpoonup v_0$ and $Lv_\varepsilon + g(v_\varepsilon) \to 0$, where $\rightharpoonup$ stands for the weak convergence in H and the convergent subsequence is denoted v_ε again.

In the above argument the element v can be an arbitrary element in the set $N_F := \{v : Lv + g(v) = 0\}$. Thus, we have proved that $||v_0|| \leq ||v||$ for all $v \in N_F$.

Let us prove that $L(v_0) + g(v_0) = 0$, i.e., $v_0 \in N_F$.

Assume first that $v_0 \in D(L)$. We prove this assumption later. The monotonicity of $L + g$ yields:

$$(L(v_\varepsilon) + g(v_\varepsilon) + \varepsilon v_\varepsilon - L(v_0 - tz) - g(v_0 - tz) - \varepsilon(v_0 - tz), v_\varepsilon - v_0 + tz) \geq 0, \qquad (2.5)$$

where $t > 0$, and $z \in D(L)$ is arbitrary. Let $\varepsilon \to 0$ in (2.5). Then, using $v_\varepsilon \rightharpoonup v_0$ and $Lv_\varepsilon + g(v_\varepsilon) \to 0$, one gets:

$$(-L(v_0 - tz) - g(v_0 - tz), \quad z) \geq 0 \qquad \forall z \in D(L). \qquad (2.6)$$

Let $t \to 0$ in (2.6). Then $(Lv_0 + g(v_0), z) \leq 0 \quad \forall z \in D(L)$. Since $D(L)$ is dense in H, it follows that $Lv_0 + g(v_0) = 0$, so $v_0 \in N_F$.

We have proved above that $\|v_0\| \leq \|v\|$ for any $v \in N_F$. Because N_F is a closed and convex set, as we prove below, and H is a uniformly convex space, there is a unique element $v \in N_F$ with minimal norm. Therefore, it follows that $v_0 = v$, where v is the minimal-norm solution to (1.2) and v_0 is the weak limit of v_ε.

Let us prove the strong convergence $v_\varepsilon \to v$.

We know that $v_\varepsilon \rightharpoonup v$. The inequality $\|v_\varepsilon\| \leq \|v\|$ implies

$$\|v\| \leq \liminf_{\varepsilon\to 0} \|v_\varepsilon\| \leq \limsup_{\varepsilon\to 0} \|v_\varepsilon\| \leq \|v\|.$$

Therefore $\lim_{\varepsilon\to 0} \|v_\varepsilon\| = ||v||$. Consequently one gets:

$$\lim_{\varepsilon\to 0} ||v_\varepsilon - v||^2 = \lim_{\varepsilon\to 0}[||v_\varepsilon||^2 + ||v||^2 - 2\Re(v_\varepsilon, v)] \leq 2[||v||^2 - (v, v)] = 0.$$

Thus, $v_\varepsilon \to v$, as claimed.

Since g is continuous, it follows that $g(v_\varepsilon) \to g(v)$.

Equation

$$Lv_\varepsilon + \varepsilon v_\varepsilon + g(v_\varepsilon) = 0, \qquad (2.7)$$

the inequality $\|v_\varepsilon\| \leq \|v\|$, and the relation $g(v_\varepsilon) \to g(v)$ imply $Lv_\varepsilon \to \eta := -g(v)$.

Let us prove that $v_0 \in D(L)$.

Because $v_0 = v$, it is sufficient to check that $v \in D(L)$. Since L is selfadjoint, and $Lv_\varepsilon \to \eta$, one has:

$$(\eta, \psi) = \lim_{\varepsilon\to 0}(Lv_\varepsilon, \psi) = \lim_{\varepsilon\to 0}(v_\varepsilon, L\psi) = (v, L\psi) \quad \forall\psi \in D(L).$$

Thus $v \in D(L)$ and $Lv = \eta$.

Let us finally check that N_F *is closed and convex.*

Assume $F(v_n) := Lv_n + g(v_n) = 0$, $v_n \to v$. Then $g(v_n) \to g(v)$ because g is continuous. Thus, $Lv_n \to \eta := -g(v)$. Since L is closed, the relations $v_n \to v$ and $Lv_n \to \eta$ imply $Lv = \eta = -g(v)$. So, $v \in N_F$, and consequently N_F *is closed.*

Assume that $0 < s < 1$, $v, w \in N_F$ and $\psi = sv + (1-s)w$. Let us show that $\psi \in N_F$. One has $w \in N_F$ if and only if

$$(F(z), z - w) \geq 0 \qquad \forall z \in D(L). \tag{2.8}$$

Indeed, if $F(w) = 0$, then $F(z) = F(z) - F(w)$, and (2.8) holds because F is a monotone operator. Conversely, if (2.8) holds, then take $z - w = t\eta$, $t =$ const > 0, $\eta \in D(L)$ is arbitrary, and get $F(w + t\eta), \eta) \geq 0$. Let $t \to 0$, then $(F(w), \eta) \geq 0$ $\forall \eta \in D(L)$. Thus, since $\overline{D(L)} = H$, one gets $w \in N_F$. If $\psi = sv + (1-s)w$, and $v, w \in N(F)$, then $(F(z), z - sv - (1-s)w) = s(F(z), z - v) + (1-s)(F(z), z - w) \geq 0$. *Thus,* N_F *is convex.*

Theorem 1.2 is proved. $\Box$

Remark 2.1. *Theorem 1.2 gives a theoretical framework for a study of nonlinear ill-posed operator equations* $F(u) = f$, *where* $F(u) = Lu + g(u)$, *where the operator* $F'(u)$ *is not boundedly invertible. In this short paper, we do not discuss the case when the data* f *are given with some error:* f_δ *is given in place of* f, $\|f_\delta - f\| \leq \delta$. *In this case one can use DSM for a stable solution of the equation* $F(v) = f$ *with noisy data* f_δ, *and one uses an algorithm for choosing stopping time* t_δ *such that* $\lim_{\delta\to 0}\|u_\delta(t_\delta) - v\| = 0$, *where* $u_\delta(t)$ *is the solution to* (1.3) *with* $\Phi = \Phi_\delta$ *chosen suitably (see [2]).*

References

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